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Chandrajit Bajaj

Guoliang Xu

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Chandrajit Bajaj and Guoliang Xu

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Chandrajit Bajaj
Purdue University
West Lafayette, Indiana, USA

Guoliang Xu
Purdue University
West Lafayette, Indiana, USA

$\Diamond$ Introduction $\Diamond$

Quite often geometric designers and engineers using NURBS (Rational B-splines with non-uniform knot spacing) like to have NURBS in a standard form, where the denominator polynomial has only positive coefficients(Peigl and Tiller 1987). This assumption is quite strong, but rids the curve of real poles (roots of the denominator polynomial) and gives the rational B-spline its convex hull property. In this gem, we show how to convert a smooth rational curve with no poles in the interval $[a, b]$ into a finite number of C^∞ standard NURB curve segments. We also show that for a single degree n smooth rational curve, the number of NURB segments required is bounded above by $\frac{n(n-1)}{2}$. We use this conversion algorithm as a final step in the NURBS approximation of algebraic plane and space curves(Bajaj and Xu 1992), (Bajaj and Xu 1993).

$\Diamond$ Standard NURB Representation $\Diamond$

The first step is to transform the rational function into Bernstein-Bézier (BB) form. Let

$$R(s) = [x(s),\ y(s),\ z(s)]^T / w(s)$$

be a rational space curve on the interval $[a, b]$, where $x(s), y(s), z(s)$ and $w(s)$ are polynomials of degree n. Since

$$t^i = \sum_{j=i}^{n} \frac{C_i^j}{C_i^n} B_j^n(t)$$

where

$$t = \frac{s-a}{b-a} \in [0, 1], \quad B_j^n(t) = C_i^n t^j (1-t)^{n-j}, \quad C_i^n = \frac{n!}{i!(n-i)!}$$

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we have, for any polynomial $P(s)$ of degree n

$$P(s) = \sum_{i=0}^{n} c_i t^i = \sum_{i=0}^{n} (\sum_{j=0}^{i} \frac{C_j^i}{C_j^n} c_j) B_i^n(t) = \sum_{i=0}^{n} b_i' B_i^n(t)$$

where $b_i' = \sum_{j=0}^{i} \frac{C_j^i}{C_j^n} c_j$. Therefore $R(s)$ can be expressed as a rational Bezier curve over [a,b]

$$r(s) = \sum_{i=0}^{n} w_i b_i B_i^n(t) / \sum_{i=0}^{n} w_i B_i^n(t)$$

where $w_i \in I\!R$, $b_i \in I\!R^3$ are Bézier weights and points respectively.

The second step is to transform the rational Bezier curve into the standard NURB representation, where denominator polynomial has positive coefficients. It is well known that a degree n rational Bezier curve over [a,b] is also a NURB with knots of multiplicity n at the two end points $t = a$, $t = b$, and the Bezier points become the DeBoor points. Thus all tht remains to be shown is the transformation of the denominator polynomial into one with positive coefficients.

Given a denominator polynomial $P(t) = \sum_{i=0}^{n} w_i B_i^n(t)$, $t \in [0,1]$, we wish to divide the interval [0,1] into subintervals, say, $0 = t_0 < t_1 < \ldots < t_{l+1} = 1$, such that the BB-form of $P(t)$ on each of the subintervals $P(t)|_{[t_i t_{i+1}]} = P_i(t) \to P_i\left(\frac{s-t_i}{t_{i+1}-t_i}\right) = \tilde{P}_i(s) = \sum w_j^i B_i^n(s)$ has positive coefficients. Without loss of generality we assume $P(t) > 0$ over [0,1] since $P(t)$ has no zero in [0,1]. We show how to compute the first breakpoint $t_1 = c$, for the remaining breakpoints can be computed in a similar fashion. By the subdivision formula $B_i^n(ct) = \sum_{j=0}^{n} B_i^j(c) B_j^n(t)$. We have in [0,c], ($s = ct$; $t \in [0,1]$)

$$\begin{aligned} P(s) = P(ct) &= \sum_{i=0}^{n} w_i B_i^n(ct) \\ &= \sum_{i=0}^{n} w_i \sum_{j=0}^{n} B_i^j(c) B_j^n(t) \\ &= \sum_{j=0}^{n} q_j(c) B_j^n(t) \end{aligned}$$

where $q_j(c) = \sum_{i=0}^{j} w_i B_i^j(c)$ is a degree j polynomial in BB form.

Note that the $\lim_{c \to 0} q_j(c) = w_0$. This is because $B_0^j(0) = 1$, $B_i^j(0) = 0$, $i > 0$. Therefore if we assume $P(t) > 0$ for $t \in [0,1]$ then $P(0) = w_0 > 0$. Take $c <$ min{all roots of $q_j(c)$ in $[0,1]$}. This $c > 0$ will guarantee all $q_j(c)$ are positive.

Example 1.0.1 *Figure 1 shows an example of this conversion for the denominator polynomial* $(1-x)^5 - x*(1-x)^4 + 2*x^2*(1-x)^3 + x^3*(1-x)^2 - x^4*(1-x) + 0.5*x^5$

The initial Bezier coefficients over [0,1] are

$$bb[0] = 1.000000 \quad bb[1] = -0.200000 \quad bb[2] = 0.200000$$
$$bb[3] = 0.100000 \quad bb[4] = -0.200000 \quad bb[5] = 0.500000$$

of which two coefficients are negative. The control points are plotted in Figure 1 with darkened colored dots. The above conversion yields two pieces in standard NURB form over [0,1] with 0.640072 as the breakpoint. The new coefficients of the two NURB pieces are

$$bb[0] = 1.000000 \quad bb[1] = 0.231913 \quad bb[2] = 0.119335$$
$$bb[3] = 0.111575 \quad bb[4] = 0.060781 \quad bb[5] = 0.060781$$

and

$$bb[0] = 0.060781 \quad bb[1] = 0.060781 \quad bb[2] = 0.076842$$
$$bb[3] = 0.125649 \quad bb[4] = 0.248051 \quad bb[5] = 0.500000$$

The new control points are plotted in Figure 1 with lighter colored dots. The curve itself, is of course the same.

◇ Upper Bounds ◇

Now we give an upper bound for the total number of NURB pieces required for a degree n rational curve.

Theorem. *Let* $P(x) = \sum_{i=0}^{n} w_i B_i^n(x), \deg(p) = n$, *and* $P(x) > 0$ *on* $[0, 1]$. *Then there exist*

$$0 = x_0 < x_1 < x_1 < \ldots x_\ell < x_{\ell+1} = 1 \tag{1}$$

with

$$\ell \leq \frac{n(n-1)}{2} \tag{2}$$

such that the Bernstein form of $P(x)$ *on* $[x_i, x_{i+1}]$ *has positive and monotonic coefficients.*

Proof. Let $Z = \bigcup_{i=1}^{n-1} \{x : p^{(j)}(x) = 0\}$. Then the cardinality of $Z \leq \frac{n(n-1)}{2}$. Take distinct x_i in $Z \bigcap (0, 1)$ and arrange them in increasing order, to obtain (1) and (2). Next subdivide the interval [0,1] into sub-intervals (x_i, x_{i+1}) for $(i = 0, 1, \ldots, \ell)$, such that $p^{(j)}(x)$ has no zero in (x_i, x_{i+1}) for $j = 0, 1, \ldots, n-1$. Let $q_i(t) := P(x_i(1-t) + x_{i+1}t)$. Then $\frac{d^j q_i(t)}{dt^j} = \frac{d^j P(x)}{dx^j}(x_{i+1} - x_i)^j$. Hence $q_i^{(j)}(t)$ has no zero in (0,1) for $j = 0, 1, \ldots, n-1$. Further, $q_i^{(0)}(t)$ has no zero on [0,1] by the earlier assumption.

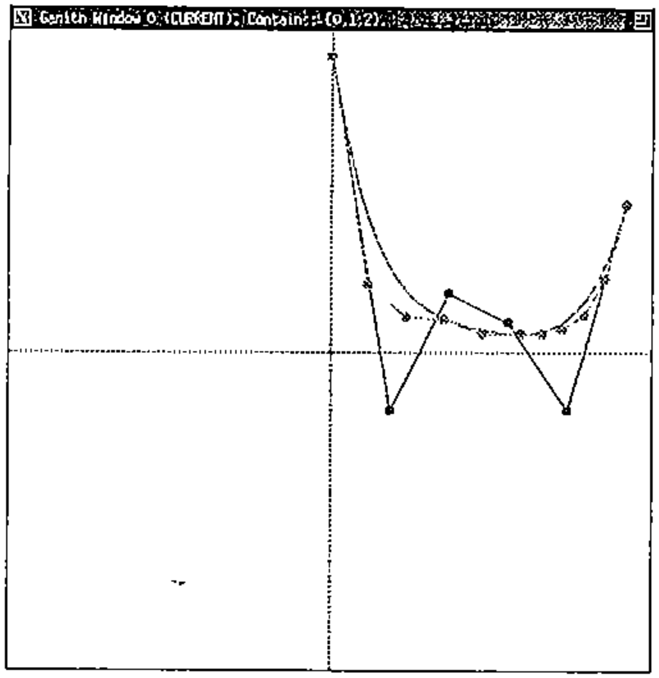

Figure 1. Denominator Polynomial with Positive Bezier Coefficients

Now we prove that the Bernstein form $q_i(t) = \sum_{j=0}^{n} w_j^{(i)} B_j^n(t), i = 0, 1, \ldots, \ell$ has positive and monotonic coefficients. In fact we prove a more general conclusion.

Lemma. *If $q(t)$ is a polynomial of degree n, and $q^{(j)}(t)$ has no zero in the open interval (0,1) for $j = 0, 1, \ldots, n$, then the coefficients of BB form representation $q^{(j)}(t) = \sum_{i=0}^{n-j} w_i^{(j)} B_i^{n-j}(t)$ are monotonic and have the same sign for any fixed j.*

Proof. We prove this fact by induction. For $j = n$, $q^{(j)}(t)$ is a nonzero constant, the required conclusion is obviously true. In general, suppose the Lemma is true for $j+1$, then for j we have since $q^{(j)}(t) = \sum_{i=0}^{n-j} w_i^{(j)} B_i^{n-j}(t)$

$$\begin{aligned} q^{(j+1)}(t) &= \sum_{i=0}^{n-j-1} w_i^{(j+1)} B_i^{n-j-1}(t) \\ &= \sum_{i=0}^{n-j-1} \Delta w_i^{(j)} B_i^{n-j-1}(t). \end{aligned}$$

where $\Delta w_i^{(j)} = w_{i+1}^{(j)} - w_i^{(j)}$, i.e., $w_{i+1}^{(j)} - w_i^{(j)} = w_i^{(j+1)}$. Since $w_i^{(j+1)}$ does not change sign, hence $w_i^{(j)}$ is monotonic. But $w_0^{(j)} = q^{(j)}(0)$ and $w_{n-j}^{(j)} = q^{(j)}(1)$ have the same sign. Hence $w_i^{(j)}$ has the same sign and the induction is complete.

Back to the proof of the theorem. We know from the above Lemma that the coefficients $w_j^{(i)}$ of $q_i(t)$ are monotonic for fixed i. Hence they are positive since $w_0^{(i)}$ and $w_n^{(i)}$ are positive.

It should be noted that the partition given in the Theorem guarantees not only positivity but also monotonicity of coefficients. This is often important because this stronger condition on the coefficients prevents the standard NURBS representation from having very small positive denominator coefficients.

◇ Psuedo Code of the Algorithm ◇

The algorithm in psuedo code for converting a polynomial into positive coefficients BB form is as follows:

$k \longleftarrow 0, \quad t_0 \longleftarrow 0,$
Transform $P(s)$ to BB form $\sum w_i^{(k)} B_i^n$ on [0,1]
while ($t_k < 1$)
begin
  if (all $w_i^{(k)} > 0$)
    begin
      $t_{k+1} \longleftarrow 1,$

```
            l ⟵ k,
            k ⟵ k + 1,
        end
    else
        begin
            compute the first breakpoint c of ∑ w_i^(k) B_i^n
            t_{k+1} ⟵ t_k + (1 − t_k)c,
            Transform ∑ w_i^(0) B_i^n on [t_{k+1}, 1] to BB form ∑ w_i^(k+1) B_i^n on [0,1]
            k ⟵ k + 1,
        endelse
    endif
endwhile
return(l, t_0, ..., t_{l+1})
```

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